位相欠陥周りのゼロモードの再考察

Revisiting the Role of Zero Modes around Topological Solitons

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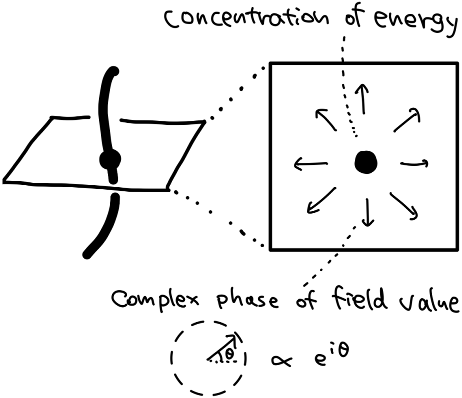
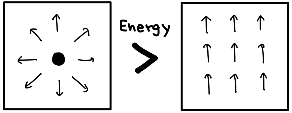
What is “Topological Soliton”?

Example: **(Cosmic) String**



(String made of complex scalar with $\mathrm{U}(1)$ symmetry)

- **Stationary point** of the energy (or the action).
- Not a global minimum → Unstable?



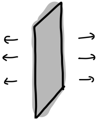
- Stable due to “**topological stability**”:

  Winding of the field around the string cannot be **deformed continuously**.

Variety of Topolocigal Solitons

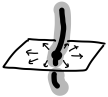
Topological solitons with various dimensionality exist.

- **Domain wall**



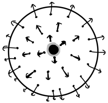
space **2**-dim + time **1**-dim

- **String**



space **1**-dim + time **1**-dim

- **Monopole**



space **0**-dim + time **1**-dim

- **Instanton**

space **0**-dim + time **0**-dim

Zero Modes around Solitons

Zero mode: Field oscillation $\phi^{(0)}$ which costs no energy (or action).

$$E[\phi + (\mathrm{const}) \times \phi^{(0)}] = E[\phi]$$

(ϕ: soliton configuration, $\phi^{(0)}$: zero mode)

Localized (decreasing at far away from the soliton) zero modes are important.

(No such mode around trivial vacuum because their spatial dependence costs energy.)

- **Boson localized zero modes**:
  Transformation of the soliton which cost no energy (translation, rotation, ...).

- **Fermion localized zero modes**:
  There often exist zero modes of fermions interacting with the soliton.

$$E[\phi, \psi \propto \psi^{(0)}] = E[\phi, \psi = 0]$$

Impact of Fermion Zero Modes

- **Fermion zero modes prevent instanton effect**:
  Path integral around the soliton vanishes due to fermion zero modes.

$$Z \propto \int \mathcal{D}\psi \, e^{-E[\psi]} \propto \int \mathrm{d}(\text{zero mode}) \times (\text{const}) = 0 \, .$$

Instanton effects on physics vanishes when fermion zero modes are present.

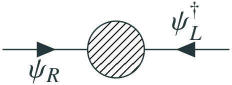
- **Superconduction along string**:
  Fermion zero modes localized in (x, y) + String direction z + Time t
  $\rightarrow$ Massless fermion along the string.

Superconducting string

Fermion Zero Modes around Instantons

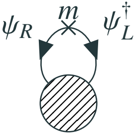
Effects of fermion zero modes are captured by fermion lines emanating from instanton.

- Case 1: $\mathrm{SU}(N)$ gauge theory with a **massless** N-plet Dirac fermion $\Psi = (\psi_L, \psi_R)$.



There exist fermion zero modes.

- Case 2: $\mathrm{SU}(N)$ gauge theory with a **massive** N-plet Dirac fermion $\Psi = (\psi_L, \psi_R)$.



Due to the mass, zero modes in Case-1 are not zero modes now.

$$E[A, \Psi \propto \Psi^{(0)}] \neq E[A]$$

(A: instanton configuration of gauge field,
$\Psi^{(0)}$: zero mode in massless case)

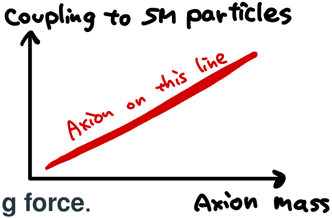
Our Work

with M. Ibe, S. Shirai and K. Watanabe

Strong CP Problem

One of free parameters of Standard Model, θ_{QCD}, is **very near to a special point of the theory**. (Strong force has CP symmetry when $\theta_{\text{QCD}} = 0$. Why $|\theta_{\text{QCD}}| \lesssim 10^{-10}$?)

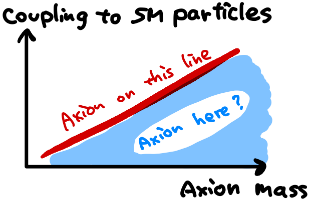
- **Axion**: attractive solution to the strong CP problem.
- Axion mass is **predicted in a model-independent way**.
  1. Axion mass is dynamically generated ($\supset$ **instantons**).
  2. To align $\theta_{\text{QCD}} \to 0$, axion dynamics is **dominated by strong force**.
  3. Mass is basically **irrelevant to small-scale physics**.



Axion Mass Enhancement by Small-scale Physics?

- Axion mass is **enhanced by instantons in hidden gauge sector** in a toy model.

  [P. Agrawal and K. Howe (2017), C. Csáki, M. Ruhdorfer and Y,Shirman (2020)]



Q. Axion mass enhancement in theoretically "nice" models?

→ Impact on search strategy and relaxation of theoretical subtlety.

Our target: A model by M. Redi and R. Sato (2016)

The model is partly **similar** to the toy model of axion mass enhancement.

→ Naively, axion mass enhancement.

Result: **NO axion mass enhancement** due to a symmetry arising simultaneously when solving the strong CP problem in a "theoretically-nice" ways.

(**Fermion zero modes** from instantons cannot be closed due to the symmetry!)

Conclusions

- Topological solitons are "**topologically stable**."
  $\rightarrow$ Domain walls, strings, monopoles, instantons, . . .

- There exist **boson / fermion zero modes** around solitons.

- Fermion zero modes have drastic impacts.
  $\rightarrow$ Superconductivity along string, Suppression of instanton effects, . . .

- Application: We have found that **axion mass is not enhnaced** in a theoretically "nice" model which looks similar to **the toy model in which axion mass is enhanced**.

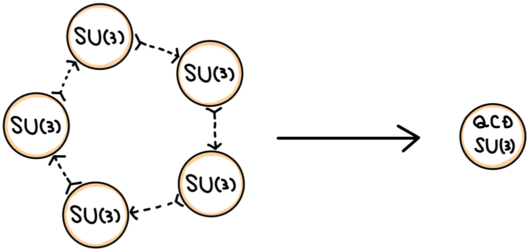
BACKUP

<u>**Axion Mass Enhancement?**</u>: →Search strategy. Relaxing a theoretical subtlety.

<u>Product group model</u>: Gauge $[\mathrm{SU}(3)]^{n_s}$ symmetry with n_s axions (by hand).

$$\mathcal{L} = \sum_{i=1}^{n_s} \left[-\frac{1}{4} F_i F_i + \left(\theta_i + \frac{a_i}{f_i} \right) \frac{g_i^2}{32\pi^2} F_i \tilde{F}_i \right] + \text{(scalars for symmetry breaking)}$$

Bi-fundamental scalars break gauge symmetry: $[\mathrm{SU}(3)]^{n_s} \to \mathrm{SU}(3)_{\mathrm{QCD}}$



Axion Mass Enhancement?

1. Small instanton effects from each $\mathrm{SU}(3)_i$: $\Delta V(a_i) \propto -v^4 \exp\left(-\frac{8\pi^2}{g_i^2(v)}\right) \cos\left(\frac{a_i}{f_i}\right)$

2. Matching of couplings: $\frac{1}{g_{\mathrm{QCD}}^2(v)} = \frac{1}{g_1^2(v)} + \ldots + \frac{1}{g_{n_s}^2(v)}$

Axion mass enhancement by large couplings of $\mathrm{SU}(3)_i$.

[Agrawal & Howe (2017), Csaki et al (2020)]

An Example: The Simplest Composite Axion Model [Choi & Kim (1985)]

New fermions

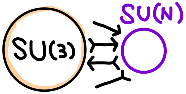
	$SU(N)$	$SU(3)_{QCD}$
ψ_1	$\mathbf{N}$	$\bar{\mathbf{3}}$
ψ_1'	$\mathbf{N}$	$\mathbf{1}$
ψ_2	$\bar{\mathbf{N}}$	$\mathbf{3}$
ψ_2'	$\bar{\mathbf{N}}$	$\mathbf{1}$

Maximal flavor symmetry for (3+1)-pairs of $(N, \bar{N})$:

(in vanishing coupling limit of $SU(3)_{QCD}$)

$$\mathbf{U(4)}_N \times \mathbf{U(4)}_{\bar{N}} = SU(4)_V \times U(1)_V \times SU(4)_A \times U(1)_A$$

$$SU(4)_V \supset \mathbf{SU(3)_{QCD}}, \qquad U(1)_A \supset \mathbf{U(1)_{PQ}}$$



PQ-breaking scale = $SU(N)$ dynamical scale

"Dimensional transmutation"

An Example: The Simplest Composite Axion Model [Choi & Kim (1985)]

New fermions

	$SU(N)$	$SU(3)_{QCD}$
ψ_1	**N**	$\bar{\mathbf{3}}$
ψ'_1	**N**	**1**
ψ_2	$\bar{\mathbf{N}}$	**3**
ψ'_2	$\bar{\mathbf{N}}$	**1**

More on Symmetries

$$U(4)_N \times U(4)_{\bar{N}} = SU(4)_V \times U(1)_V \times SU(4)_A \times U(1)_A$$

$$SU(4)_V \supset \mathbf{SU(3)_{QCD}}, \qquad SU(4)_A \supset \mathbf{U(1)_{PQ}}$$

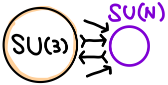
- $U(1)_A$ is $SU(N)$-anomalous, nullifying $SU(N)$ θ-angle.
- Spontaneously broken $SU(4)_A$ yields 15 would-be Nambu-Goldstone bosons.

QCD colors: $\mathbf{15} = \mathbf{8} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1}$

Model

The Simple Model

	SU(N)	SU(3)$_{\text{QCD}}$
ψ_1	$\mathbf{N}$	$\overline{\mathbf{3}}$
ψ'_1	$\mathbf{N}$	$\mathbf{1}$
ψ_2	$\overline{\mathbf{N}}$	$\mathbf{3}$
ψ'_2	$\overline{\mathbf{N}}$	$\mathbf{1}$

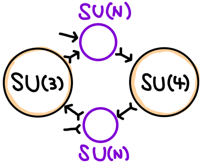


U(1)$_{\text{PQ}}$ is imposed **by hand**.

A Model Addressing the Quality Problem

[M. Redi & R. Sato (2016)]

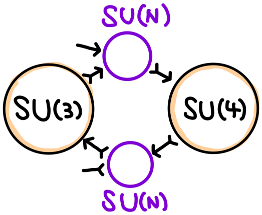
	SU(N)$_{\text{ST}\,2}$	SU(3)$_{\text{W}}$	SU(N)$_{\text{ST}\,1}$	SU(4)$_{\text{W}}$
ψ_1	$\mathbf{N}$	$\overline{\mathbf{3}}$		
ψ'_1	$\mathbf{N}$	$\mathbf{1}$		
ψ_2		$\mathbf{3}$	$\overline{\mathbf{N}}$	
ψ'_2		$\mathbf{1}$	$\overline{\mathbf{N}}$	
ψ_3			$\mathbf{N}$	$\overline{\mathbf{4}}$
ψ_4	$\overline{\mathbf{N}}$			$\mathbf{4}$



U(1)$_{\text{PQ}}$ is **accidental**.

Model – Symmetries

	$SU(N)_{ST2}$	$SU(3)_W$	$SU(N)_{ST1}$	$SU(4)_W$
ψ_1	$\mathbf{N}$	$\bar{\mathbf{3}}$		
ψ_1'	$\mathbf{N}$	$\mathbf{1}$		
ψ_2		$\mathbf{3}$	$\bar{\mathbf{N}}$	
ψ_2'		$\mathbf{1}$	$\bar{\mathbf{N}}$	
ψ_3			$\mathbf{N}$	$\bar{\mathbf{4}}$
ψ_4	$\bar{\mathbf{N}}$			$\mathbf{4}$



Flavor symmetry of $SU(N)_{ST1} \times SU(N)_{ST2}$ dynamics:

$$U(4)_1^N \times U(4)_1^{\bar{N}} \times U(4)_2^N \times U(4)_2^{\bar{N}} \supset SU(3)_W \times SU(4)_W \times [\mathbf{U(1)}]^6$$

1. Two U(1)'s are anomalous w.r.t. $SU(N)_{ST1} \times SU(N)_{ST2}$, aligning their θ angles.
2. The other $[\mathbf{U(1)}]^4$: $U(1)_{PQ}^{(SSB)} \times U(1)_1$ $\times [U(1)]^2$ (:anomalous)

Model – Spontaneous Breaking

Fermion Condensations

$\mathrm{SU}(3)_{\mathrm{W}}$, $\mathrm{SU}(4)_{\mathrm{W}}$ couplings $\to 0$.

Then, $[\mathrm{SU}(N)_{\mathrm{ST}}]^2$: independent vector-like theories.

$$\mathrm{U}(4)_1^N \times \mathrm{U}(4)_1^{\bar{N}} \to [\mathrm{U}(4) \text{ subgroup}]$$
$$\mathrm{U}(4)_2^N \times \mathrm{U}(4)_2^{\bar{N}} \to [\mathrm{U}(4) \text{ subgroup}]$$

	$\mathrm{SU}(N)_{\mathrm{ST}2}$	$\mathrm{SU}(3)_{\mathrm{W}}$	$\mathrm{SU}(N)_{\mathrm{ST}1}$	$\mathrm{SU}(4)_{\mathrm{W}}$	$\mathrm{U}(1)_{\mathrm{PQ}}$
ψ_1	**N**	$\bar{\mathbf{3}}$			1
ψ_1'	**N**	**1**			−3
ψ_2		**3**	$\bar{\mathbf{N}}$		1
ψ_2'		**1**	$\bar{\mathbf{N}}$		−3
ψ_3			**N**	$\bar{\mathbf{4}}$	0
ψ_4	$\bar{\mathbf{N}}$			**4**	0

Gauge symmetries: $\mathrm{SU}(3)_{\mathrm{W}} \times \mathrm{SU}(4)_{\mathrm{W}} \to \mathrm{SU}(3)_{\mathrm{QCD}}$: diagonal subgroup

Global symmetries: Broken $[\mathrm{U}(4)]^2 = [\mathrm{U}(1)]^2 \times [\mathrm{SU}(4)]^2 \supset \mathrm{U}(1)_{\mathrm{PQ}}$

($[\mathrm{SU}(N)_{\mathrm{ST}}]^2$-anomalous $[\mathrm{U}(1)]^2$)

+ (15 massive gauge bosons)

+ $(\mathbf{15} = \mathbf{8} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1})$

Accidental $[\mathrm{U}(1)]^4$ Global Symmetry

$$\mathbf{U(4)}_{\mathbf{1}}^{N} \times \mathbf{U(4)}_{\mathbf{1}}^{\bar{N}} \times \mathbf{U(4)}_{\mathbf{2}}^{N} \times \mathbf{U(4)}_{\mathbf{2}}^{\bar{N}} \supset \mathrm{SU(3)_W} \times \mathrm{SU(4)_W} \times \mathbf{U(1)_{PQ}} \times \mathbf{U(1)_1} \times \mathbf{U(1)_2} \times \mathbf{U(1)_3}$$

(anomalous)

	$\mathrm{SU}(N)_{\mathrm{ST2}}$	$\mathrm{SU(3)_W}$	$\mathrm{SU}(N)_{\mathrm{ST1}}$	$\mathrm{SU(4)_W}$	$\mathrm{U(1)}_{\mathrm{PQ}}^{(\mathrm{SSB})}$	$\mathrm{U(1)_1}$	$\mathrm{U(1)_2}$	$\mathrm{U(1)_3}$
ψ_1	$\mathbf{N}$	$\bar{\mathbf{3}}$			1	1	1	1
ψ_1'	$\mathbf{N}$	$\mathbf{1}$			−3	1	−3	1
ψ_2		$\mathbf{3}$	$\bar{\mathbf{N}}$		1	1	−1	−1
ψ_2'		$\mathbf{1}$	$\bar{\mathbf{N}}$		−3	1	3	−1
ψ_3			$\mathbf{N}$	$\bar{\mathbf{4}}$	0	−1	0	1
ψ_4	$\bar{\mathbf{N}}$			$\mathbf{4}$	0	−1	0	−1

Two θ-angles of $\mathrm{SU(3)_W}$ and $\mathrm{SU(4)_W}$ $\leftarrow$ “nullified” by $\mathbf{U(1)_{PQ}}$ and $\mathbf{U(1)_1}$.

Axion Mass Enhancement?

Hidden instantons:

$\mathrm{SU}(3)_{\mathrm{W}}$ and $\mathrm{SU}(4)_{\mathrm{W}}$ hidden small instantons:

$$\propto \Lambda^4 \exp\left(-\frac{8\pi^2}{g^2_{\mathrm{SU}(3)_{\mathrm{W}}}(\Lambda)}\right) \quad \text{and} \quad \propto \Lambda^4 \exp\left(-\frac{8\pi^2}{g^2_{\mathrm{SU}(4)_{\mathrm{W}}}(\Lambda)}\right).$$

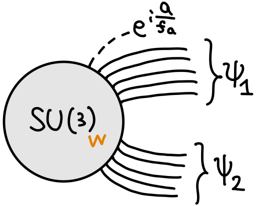
PQ symmetry: anomalous with respect to $\mathrm{SU}(3)_{\mathrm{W}}$

$\longrightarrow$ Hidden (small) instantons: milder supperssion **with coupling stronger than QCD**. $\left(\frac{1}{g^2_{\mathrm{QCD}}(\Lambda)} = \frac{1}{g^2_{\mathrm{SU}(3)_{\mathrm{W}}}(\Lambda)} + \frac{1}{g^2_{\mathrm{SU}(4)_{\mathrm{W}}}(\Lambda)}\right.$ permits $\left. g_{\mathrm{SU}(3)_{\mathrm{W}}}(\Lambda) \gg g_{\mathrm{QCD}}(\Lambda)\right)$

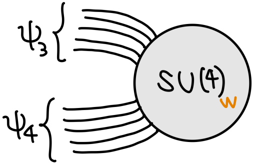
Question: Axion mass enhancement by $\mathrm{SU}(3)_{\mathrm{W}}$ hidden instanton?

't Hooft Vertex: ['t Hooft (1976)]

Effect of fermion zero modes around instantons are captured by " 't Hooft vertex"



$(1, 0)$ instanton



$(0, 1)$ instanton

	$SU(N)_{ST2}$	$SU(3)_W$	$SU(N)_{ST1}$	$SU(4)_W$
ψ_1	$\mathbf{N}$	$\overline{\mathbf{3}}$		
ψ_1'	$\mathbf{N}$	$\mathbf{1}$		
ψ_2		$\mathbf{3}$	$\overline{\mathbf{N}}$	
ψ_2'		$\mathbf{1}$	$\overline{\mathbf{N}}$	
ψ_3			$\mathbf{N}$	$\overline{\mathbf{4}}$
ψ_4	$\overline{\mathbf{N}}$			$\mathbf{4}$

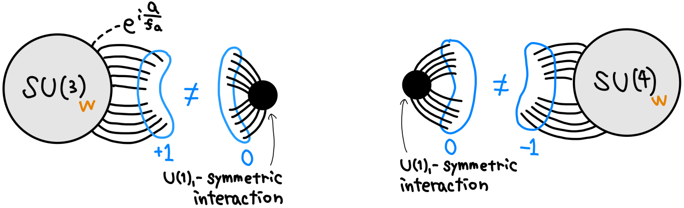
Axion Mass Enhancement?

$\mathrm{U}(1)_1$ charge of fermions around $(1,0)$ or $(0,1)$ Instantons:

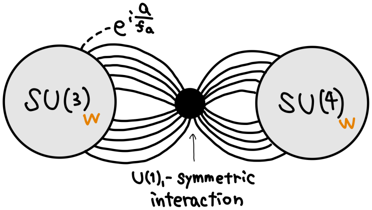
	$SU(N)_{ST2}$	$SU(3)_W$	$SU(N)_{ST1}$	$SU(4)_W$	$U(1)_{PQ}^{(SSB)}$	$U(1)_1$	$U(1)_2$	$U(1)_3$
ψ_1	**N**	$\bar{\mathbf{3}}$			1	**+1**	1	1
ψ'_1	**N**	**1**			-3	**+1**	-3	1
ψ_2		**3**	$\bar{\mathbf{N}}$		1	**+1**	-1	-1
ψ'_2		**1**	$\bar{\mathbf{N}}$		-3	**+1**	3	-1
ψ_3			**N**	$\bar{\mathbf{4}}$	0	**-1**	0	1
ψ_4	$\bar{\mathbf{N}}$			**4**	0	**-1**	0	-1

Axion Mass Enhancement?

Fermion legs cannot be closed around a single $(1,0)$ or $(0,1)$ instanton,



while their pair with $\mathrm{U}(1)_1$-symmetric interactions generated by strong dynamics.



No Axion Mass Enhancement!

Small instanton effects are always from “pairs”

A pair of $(1,0)$ and $(0,1)$ instanton:

$$\exp\left(-\frac{8\pi^2}{g^2_{SU(3)_W}(\Lambda)}\right)\times\exp\left(-\frac{8\pi^2}{g^2_{SU(4)_W}(\Lambda)}\right)=\exp\left(-8\pi^2\left[\frac{1}{g^2_{SU(3)_W}(\Lambda)}+\frac{1}{g^2_{SU(4)_W}(\Lambda)}\right]\right)$$

$$=\exp\left(-\frac{8\pi^2}{g^2_{QCD}(\Lambda)}\right)$$

- Matching of couplings: $\frac{1}{g^2_{QCD}(\Lambda)}=\frac{1}{g^2_{SU(3)_W}(\Lambda)}+\frac{1}{g^2_{SU(4)_W}(\Lambda)}$

→ **No axion mass enhancement by large coupling of $SU(3)_W$.**

(Even when $g_{SU(3)_W}\gg g_{QCD}$, effects are accompanied by $g_{SU(4)_W}\sim g_{QCD}$ suppression.)

No Axion Mass Enhancement!: Comment on Axion Quality

Lowest-Dimensional PQ-Breaking Operator

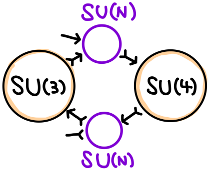
PQ-breaking, dimension-6 operator:

$$\psi_1\psi_2\psi_3\psi_4$$

Not enough to avoid **quality problem**.

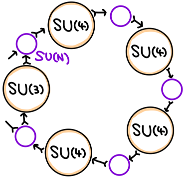
	$SU(N)_{ST2}$	$SU(3)_W$	$SU(N)_{ST1}$	$SU(4)_W$
ψ_1	$\mathbf{N}$	$\bar{\mathbf{3}}$		
ψ_1'	$\mathbf{N}$	$\mathbf{1}$		
ψ_2		$\mathbf{3}$	$\bar{\mathbf{N}}$	
ψ_2'		$\mathbf{1}$	$\bar{\mathbf{N}}$	
ψ_3			$\mathbf{N}$	$\bar{\mathbf{4}}$
ψ_4	$\bar{\mathbf{N}}$			$\mathbf{4}$

$\rightarrow$ In Redi and Sato model, **improvement of axion quality** is easily possible.



$n_s = 2$

PQ breaking dimension: $4 + d \geq 6$



$n_s = 5$

PQ breaking dimension: $4 + d \geq 15$

No axion mass enhancement also for $n_s \geq 3$.

Quality Problem and Larger Model

Lowest-Dimensional PQ-Breaking Operator

PQ-breaking, dimension-6 operator:

$$\psi_1\psi_2\psi_3\psi_4$$

Not enough to avoid **quality problem**.

	$SU(N)_{ST2}$	$SU(3)_W$	$SU(N)_{ST1}$	$SU(4)_W$
ψ_1	**N**	$\bar{\mathbf{3}}$		
ψ'_1	**N**	**1**		
ψ_2		**3**	$\bar{\mathbf{N}}$	
ψ'_2		**1**	$\bar{\mathbf{N}}$	
ψ_3			**N**	$\bar{\mathbf{4}}$
ψ_4	$\bar{\mathbf{N}}$			**4**

In larger model with $[SU(N)_{ST}]^{n_s}\times SU(3)_W\times[SU(4)_W]^{n_s-1}$ symmetry, dimension-$3n_s$ is the lowest-dimension PQ breaking.

$(n_s = 3 \ \rightarrow)$

	$SU(N)_{ST3}$	$SU(3)_W$	$SU(N)_{ST1}$	$SU(4)_{W1}$	$SU(N)_{ST2}$	$SU(4)_{W2}$
ψ_1	**N**	$\bar{\mathbf{3}}$				
ψ'_1	**N**	**1**				
ψ_2		**3**	$\bar{\mathbf{N}}$			
ψ'_2		**1**	$\bar{\mathbf{N}}$			
ψ_3			**N**	$\bar{\mathbf{4}}$		
ψ_4				**4**	$\bar{\mathbf{N}}$	
ψ_5					**N**	$\bar{\mathbf{4}}$
ψ_6	$\bar{\mathbf{N}}$					**4**

We find **no enhancement similarly in larger models with $n_s \geq 3$**

Larger Model ($n_s = 3$)

	$SU(N)_{ST3}$	$SU(3)_W$	$SU(N)_{ST1}$	$SU(4)_{W1}$	$SU(N)_{ST2}$	$SU(4)_{W2}$
ψ_1	**N**	$\bar{\mathbf{3}}$				
ψ_1'	**N**	**1**				
ψ_2		**3**	$\bar{\mathbf{N}}$			
ψ_2'		**1**	$\bar{\mathbf{N}}$			
ψ_3			**N**	$\bar{\mathbf{4}}$		
ψ_4				**4**	$\bar{\mathbf{N}}$	
ψ_5					**N**	$\bar{\mathbf{4}}$
ψ_6	$\bar{\mathbf{N}}$					**4**

	$U(1)_{PQ}^{(SSB)}$	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_4$
ψ_1	1	1	1	1	0
ψ_1'	−3	1	−3	1	0
ψ_2	1	1	−1	−1	0
ψ_2'	−3	1	3	−1	0
ψ_3	0	−1	0	1	0
ψ_4	0	0	0	−1	−1
ψ_5	0	0	0	1	1
ψ_6	0	−1	0	−1	0

- Only $U(1)_{PQ}$ is spontaneously broken, also for larger n.
- Additional $n-2$ anomalous (and unbroken) $U(1)$s, cancelling the additional θ angles.

Axion Mass Enhancement?

Another Explanation: Directly from symmetry, without relying on 't Hooft vertex.

Vacuum amplitude with fixed axion field value a and background $\mathrm{SU}(3)_{\mathrm{W}}$ and $\mathrm{SU}(4)_{\mathrm{W}}$ gauge field:

$$W(a)|_{m,n} = \int \prod \mathcal{D}A_{\mathrm{ST}} \mathcal{D}\psi^\dagger \mathcal{D}\psi \, e^{-S[\psi, A_{\mathrm{ST}}, a]}$$

- m, n are winding number for each sectors.
- The amplitude = contribution to the axion potential.

We can redefine (rename) the fermions in path integral by $\mathrm{U}(1)_1$ rotation $e^{i\alpha}$.

$\rightarrow W(a)|_{m,n}$ changes its phase by anomaly, without shifting the axion a.

$$W(a)|_{m,n} = \exp\left[2i\alpha(m-n)\right] W(a)|_{m,n}.$$

The amplitude (effects on the axion potential) vanishes, unless $m = n$