Unitarity test of lepton mixing via energy dependence of neutrino oscillation

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Motivation

- Neutrino mass remains an open question
- Studying the lepton mixing matrix holds the key to this problem
- Testing the unitarity of the PMNS matrix is one method to explore the general lepton mixing matrix

Introduction

Introduction

Neutrino mass remains a mystery
→ Origin of the lepton mixing matrix is also a mystery

Some models (e.g. eV-scale sterile neutrino oscillation) require mixing matrix larger than 3×3

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ \vdots \end{pmatrix} = \begin{pmatrix} V_{e1} & V_{e2} & V_{e3} & \cdots \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} & \cdots \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \vdots \end{pmatrix}$$

Exploring a general lepton mixing matrix is important
→ **Unitarity Test**

Introduction

Framework of standard 3 flavor ν oscillation

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If the PMNS matrix is 3×3 unitary, there are only 4 parameters

But this parametrization always guarantees unitarity

→**Must remove this parametrization to test unitarity**

Neutrino oscillation
and
Unitarity

Neutrino oscillation

In vacuum

$$P\left(\nu_{\alpha\to\beta}(\bar{\nu}_{\alpha\to\beta})\right)$$

$$= \delta_{\alpha\beta} - 4\sum_{j>k} \mathrm{Re}\left[U_{\alpha j}U^*_{\beta j}U^*_{\alpha k}U_{\beta k}\right]\sin^2\left(\frac{\Delta m^2_{jk}}{4E}\right)$$

$$\mp 2\sum_{j>k} \mathrm{Im}\left[U_{\alpha j}U^*_{\beta j}U^*_{\alpha k}U_{\beta k}\right]\sin\left(\frac{\Delta m^2_{jk}}{2E}\right)$$

Oscillation probability
= (**Coefficient**) × (**Energy-dependent function**)

Coefficients are just matrix elements

Observing energy-dependence can extract coefficients

Neutrino oscillation and Unitarity

In vacuum, 3-generation

Up to 2nd. order in $\Delta m^2_{21}/\Delta m^2_{31}$, U_{e3}

$$P(\nu_{\mu\to e}) = 4\left|U_{\mu3}U^*_{e3}\right|^2 \sin^2(\Delta_{31}) + 4\mathrm{Re}\left[U_{\mu3}U^*_{e3}U^*_{\mu2}U_{e2}\right]\Delta_{21}\sin(2\Delta_{31})$$
$$-4\mathrm{Re}\left[U_{\mu2}U^*_{e2}U^*_{\mu1}U_{e1}\right]\Delta^2_{21} - 8\mathrm{Im}\left[U_{\mu3}U^*_{e3}U^*_{\mu2}U_{e2}\right]\Delta_{21}\sin^2(\Delta_{31})$$

$$= C_1\cdot\sin^2(\Delta_{31}) + C_2\cdot\Delta_{21}\sin(2\Delta_{31})$$
$$+C_3\cdot\Delta^2_{21} + C_4\cdot\Delta_{21}\sin^2(\Delta_{31})$$

$$\Delta_{jk}\equiv\frac{\Delta m^2_{jk}}{4E}$$

By using unitarity condition of PMNS matrix

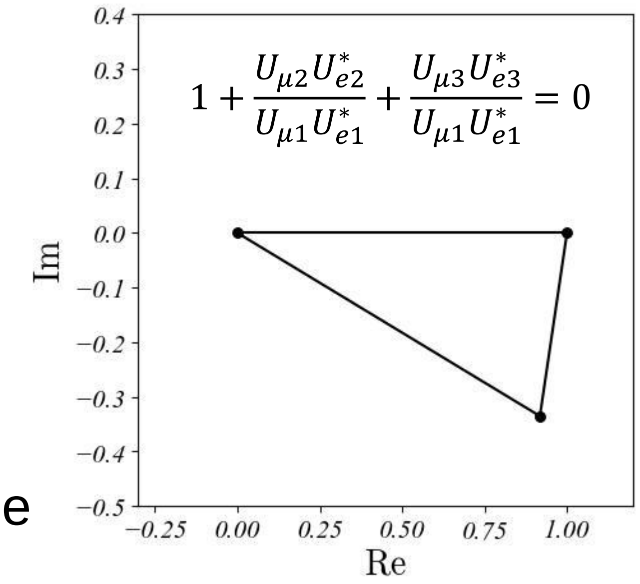
$$C_1(C_3 - C_2) = C_2^2 + \frac{C_4^2}{4}$$

This relation is Independent of parametrization

Unitarity

The unitarity condition corresponds to drawing a triangle on the complex plane

$$U_{\mu 1}U_{e1}^* + U_{\mu 2}U_{e2}^* + U_{\mu 3}U_{e3}^* = 0 \implies$$



If the unitarity condition is satisfied
→ The triangle closes as its vertices converge

Corresponding to unitarity condition of PMNS matrix

$$U_{\mu 1}U_{e1}^* + U_{\mu 2}U_{e2}^* + U_{\mu 3}U_{e3}^* = 0 \implies \xi \equiv C_1(C_3 - C_2) - C_2^2 - \frac{C_4^2}{4}$$

If unitary → $\xi = 0$
If non-unitary → $\xi \neq 0$

ξ is a new criterion of Unitarity Test independent of parametrization

Statistical analysis

Coefficient fit

Oscillation probability at a given energy bin E_j

$$P(\nu_{\mu\to e}, E_j) = C_1 \cdot B_1(E_j) + C_2 \cdot B_2(E_j) + C_3 \cdot B_3(E_j) + C_4 \cdot B_4(E_j)$$

$$B_1 = \sin^2(\Delta_{31}),\ B_2 = \Delta_{21}\sin(\Delta_{31}),$$
$$B_3 = \Delta_{21}^2,\ B_4 = \Delta_{21}\sin^2(\Delta_{31})$$

$$\chi^2 \equiv \sum_j \left[\frac{P^{\mathrm{obs}}(E_j) - \sum_{k=1}^{4} C_k \cdot B_k}{\Delta P(E_j)}\right]^2$$

Minimum is obtained by

$$\frac{d}{dC_l}\chi^2 = 0$$

Coefficient fit

χ^2 can be defined as a form of matrix and vector

$$\chi^2 \equiv \sum_j \left[\frac{P^{\text{obs}}(E_j) - \sum_{k=1}^{4} C_k \cdot B_k(E_j)}{\Delta P^{\text{obs}}(E_j)}\right]^2$$

$$= \left(\boldsymbol{P}^{\text{obs}} - \mathcal{B}\boldsymbol{C}\right)^T W \left(\boldsymbol{P}^{\text{obs}} - \mathcal{B}\boldsymbol{C}\right)$$

$$\boldsymbol{P}^{\text{obs}} \equiv \begin{pmatrix} P^{\text{obs}}(E_j) \\ \vdots \\ P^{\text{obs}}(E_n) \end{pmatrix}, \ \mathcal{B} \equiv \begin{pmatrix} B_1(E_1) & \cdots & B_4(E_1) \\ \vdots & \ddots & \vdots \\ B_1(E_n) & \cdots & B_4(E_n) \end{pmatrix}, \boldsymbol{C} \equiv \begin{pmatrix} C_1 \\ \vdots \\ C_4 \end{pmatrix}$$

$$W \equiv \text{diag}\left(\frac{1}{\left(\Delta P^{\text{obs}}(E_1)\right)^2}, \ \cdots \ , \frac{1}{\left(\Delta P^{\text{obs}}(E_n)\right)^2}\right)$$

Best fit points of coefficient $C_1 \sim C_4$

$$\frac{d}{d\boldsymbol{C}}\chi^2 = 0 \Longrightarrow \boldsymbol{C} = (\mathcal{B}^T W \mathcal{B})^{-1} \mathcal{B}^T W \boldsymbol{P}^{\text{obs}}$$

Coefficient fit

Generate 10^6 virtual-experiments
and χ^2 fit with 3-gen model

4-gen events

Generate events
by 4-gen ν oscillation
model

3-gen events

Generate events
by 3-gen ν oscillation
model

↓ ↓

Fitting 3-gen model to each events

※Probability distribution is assumed to
be binomial distribution

Reference values

3-gen reference values (arithmetic average of bfp in PDG exept for δ)

$\Delta m_{21}^2/10^{-5}$ eV	$\Delta m_{31}^2/10^{-3}$ eV	θ_{12}	θ_{13}	θ_{23}	δ
7.43	2.432	33.9°	8.49°	48.1°	270°

4-gen reference values (bfp in Parveen et al. 2025)

$\Delta m_{21}^2/10^{-5}$ eV	$\Delta m_{31}^2/10^{-3}$ eV	$\Delta m_{41}^2/10^{-3}$ eV
7.5	2.55	1.0

θ_{12}	θ_{13}	θ_{23}	θ_{14}	θ_{24}	θ_{34}	$\delta_{13}(\delta)$	δ_{24}	δ_{34}
34.3°	8.53°	49.3°	5.7°	5°	20°	-165.6°	0°	0°

In this study, we consider only Normal Mass Ordering and statistical errors.

Number of events at Hyper-Kamiokande

We assume T2HK and neutrino factory with ν_e at J-PARC

※ When observing ν from μ decay at HK, we need to identify the charges of muons generated by CC interactions
In principle, neutron tagging method can identify.

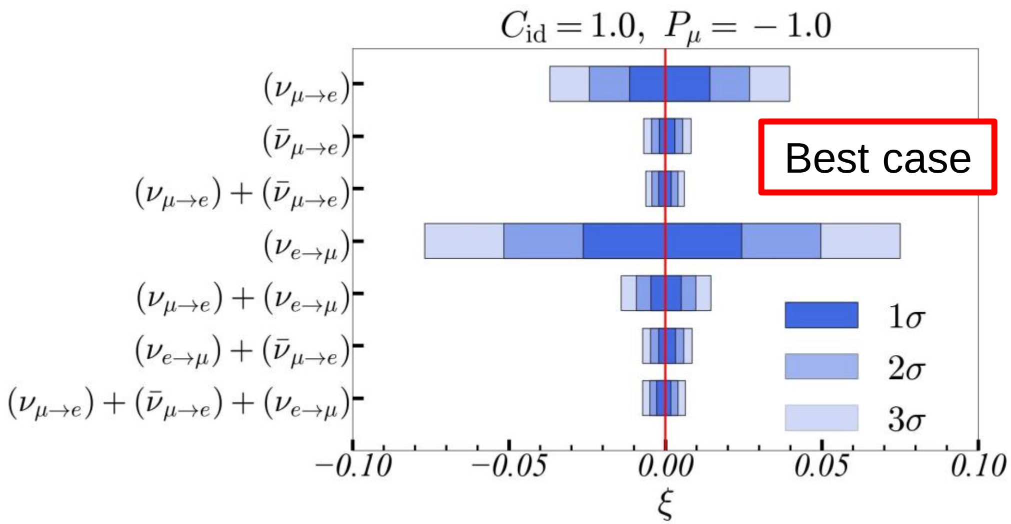
※ Muon beam polarization is also important, because neutrino flux depend on it.

In this talk, we show only 2 cases

- Charge identification efficiency and muon polarization are perfect (Best case)
- No charge identification and unpolarized muon (Worst case)

Unitarity Test

Fitting 3-gen model to 3-gen events

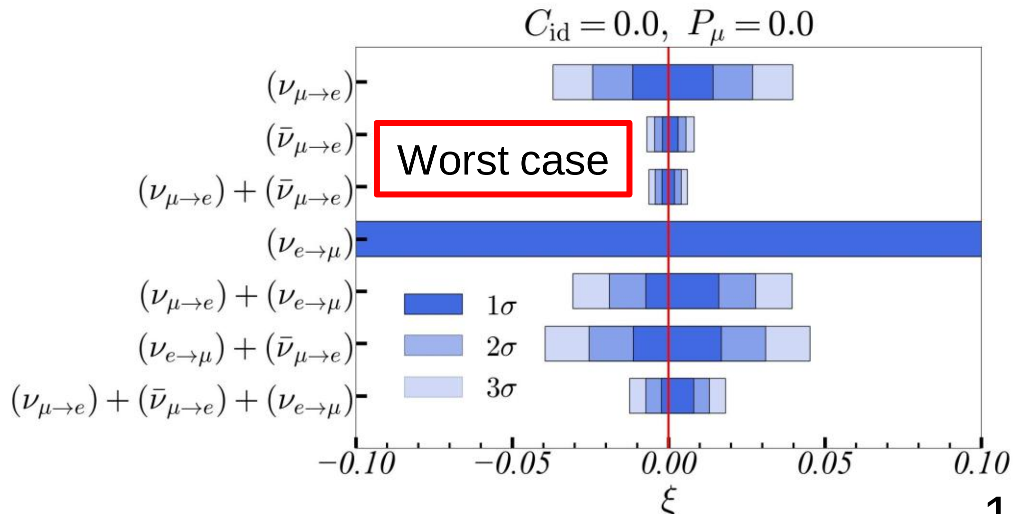


Test of consistency between 3-gen model and 3-gen events in terms of ξ

● Combining multiple channels yields a much smaller statistical error

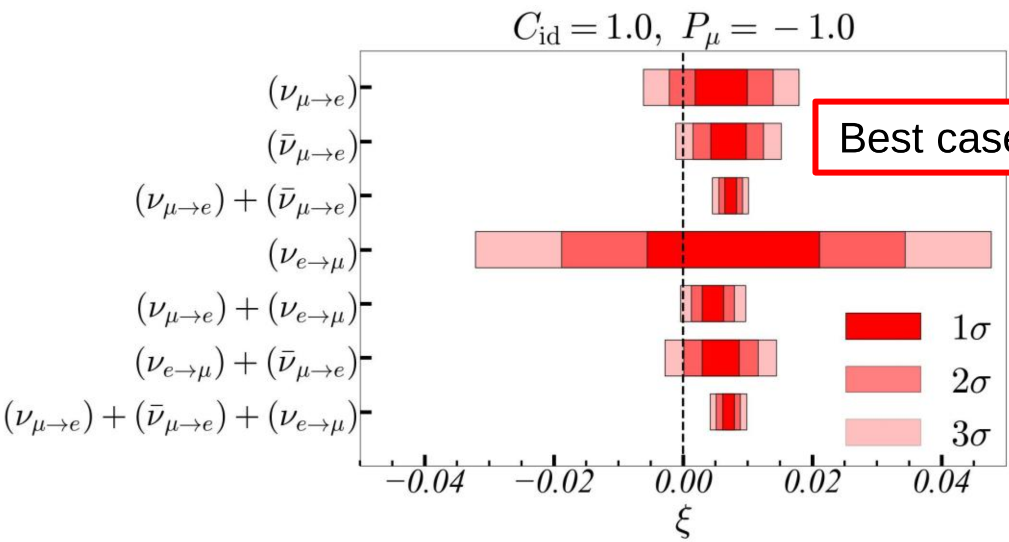
● Charge identification and muon polarization are important factor, particularly in the analysis using $(\nu_{e\to\mu})$

● All analyses are consistent with $\xi = 0$



Unitarity Test

Fitting 3-gen model to 4-gen events



Best case

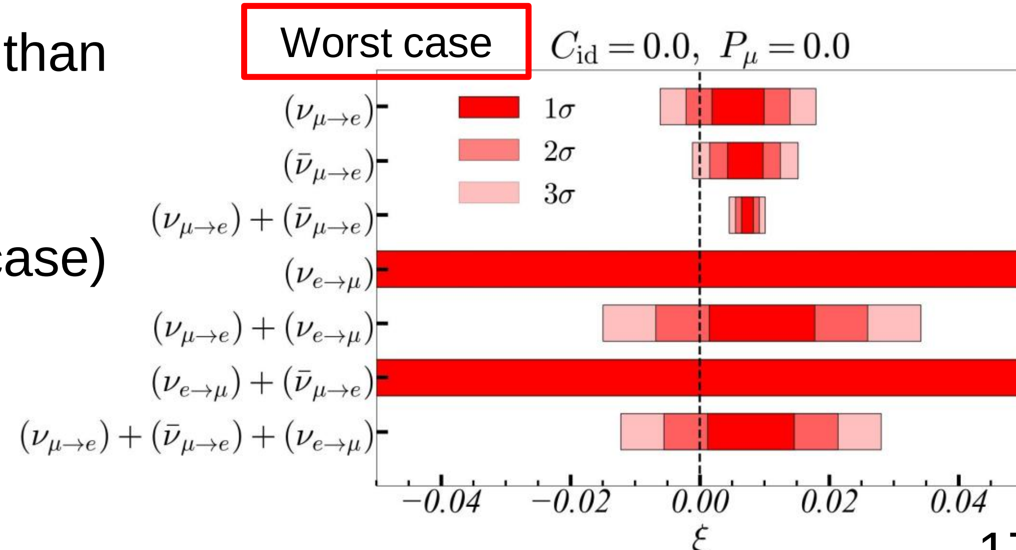
Test of inconsistency between 3-gen model and 4-gen events in terms of ξ

- Combining multiple channels yields a much smaller statistical error

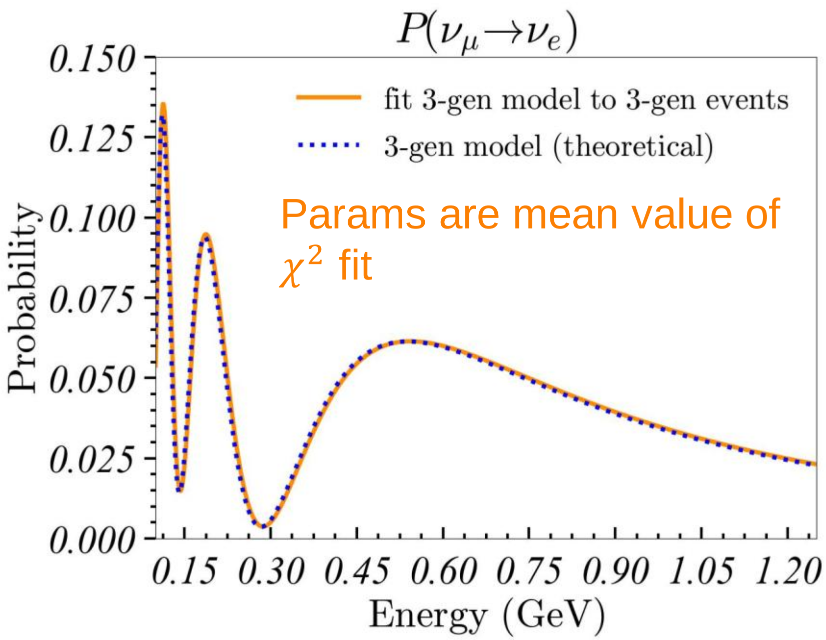
- Inconsistent with $\xi = 0$ at more than 3σ,
  - $(\nu_{\mu\to e}) + (\bar{\nu}_{\mu\to e})$
  - $(\nu_{\mu\to e}) + (\bar{\nu}_{\mu\to e}) + (\nu_{e\to\mu})$ (Best case)

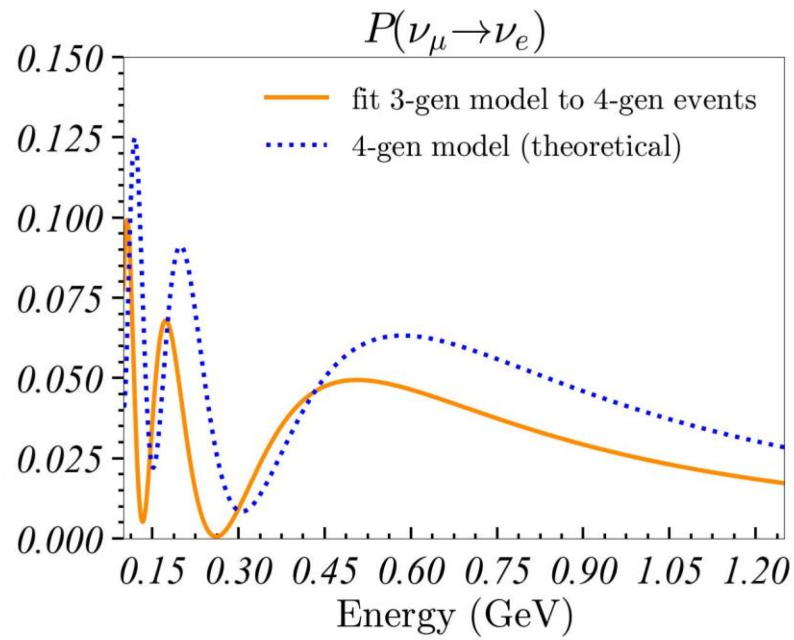
Sufficient to test the violation of unitarity !!

Worst case



Oscillation probability using the results of fit





Fitting the 3-gen model to 3-gen events (orange solid)

Consistent

Theoretical curve of the 3-gen model (blue dotted)

Fitting the 3-gen model to 4-gen events (orange solid)

Inconsistent

Theoretical curve of the 4-gen model (blue dotted)

Summary

- Studying a general lepton mixing is important to understand neutrino mass.

- One method to explore a general lepton mixing matrix is testing unitarity of it.

- We propose a method for testing unitarity that does not depend on the parameterization of the PMNS matrix, and demonstrate its application using the new criterion ξ.

- Fitting 3-gen model to 4-gen events results in a 3σ inconsistency with $\xi = 0$, in the $\left(\nu_{\mu\to e}\right) + \left(\bar{\nu}_{\mu\to e}\right)$ from T2HK and in the $\left(\nu_{\mu\to e}\right) + \left(\bar{\nu}_{\mu\to e}\right) + \left(\nu_{e\to\mu}\right)$ from T2HK+neutrino factory

- In this study, we consider neutrino oscillations in vacuum, but for a more realistic analysis, matter effects must be taken into account.

Back up

Neutrino oscillations in vacuum

Time evolution equation of neutrinos

$$i\frac{d}{dt}\begin{pmatrix}\nu_e(\bar{\nu}_e)\\ \nu_\mu(\bar{\nu}_\mu)\\ \nu_\tau(\bar{\nu}_\tau)\end{pmatrix} = \left[U^{(*)}\mathrm{diag}(0,\Delta E_{21},\Delta E_{31})U^{\dagger(T)}\right]\begin{pmatrix}\nu_e(\bar{\nu}_e)\\ \nu_\mu(\bar{\nu}_\mu)\\ \nu_\tau(\bar{\nu}_\tau)\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\nu_e\ (t)\\ \nu_\mu\ (t)\\ \nu_\tau\ (t)\end{pmatrix} = U\mathrm{diag}\left(0, e^{-i\Delta E_{21}t}, e^{-i\Delta E_{31}t}\right)U^\dagger\begin{pmatrix}\nu_e(0)\\ \nu_\mu(0)\\ \nu_\tau(0)\end{pmatrix}$$

$$P(\nu_\alpha \to \nu_\beta) = \left|A(\nu_\alpha \to \nu_\beta)\right|^2 = \left|\sum_j U_{\beta j}e^{-i\Delta E_{j1}t}U^*_{\alpha j}\right|^2$$

Oscillation probability

Probability amplitude

$$\Delta E_{jk} = E_j - E_k$$

Neutrino oscillations in matter

When the neutrino beam propagate in matter, we need to consider the matter effect

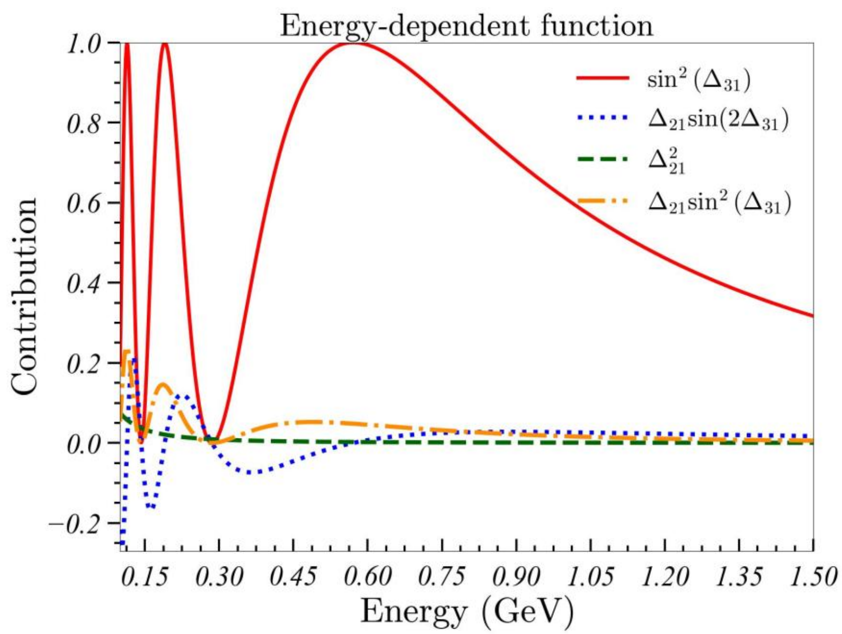
+ for ν, − for $\bar{\nu}$

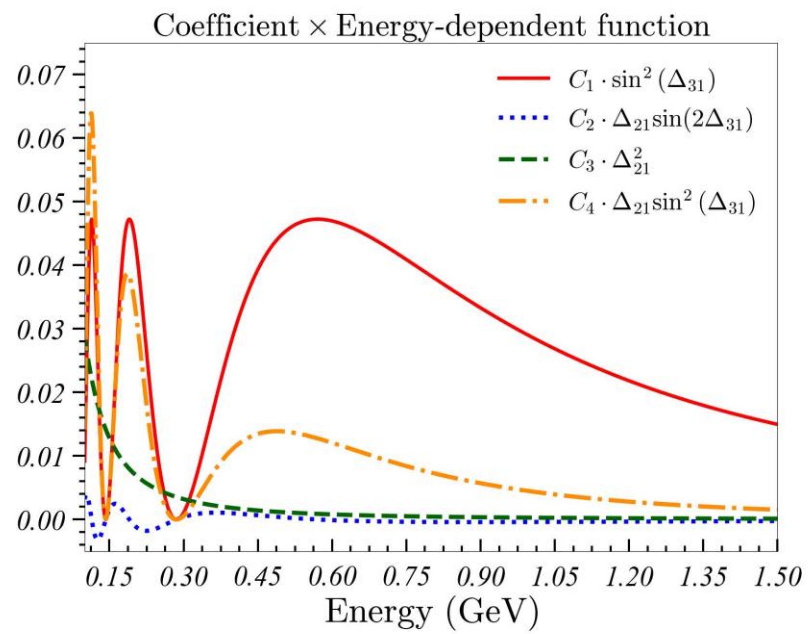
$$i\frac{d}{dt}\begin{pmatrix}\nu_e(\bar{\nu}_e)\\ \nu_\mu(\bar{\nu}_\mu)\\ \nu_\tau(\bar{\nu}_\tau)\end{pmatrix} = \left[U^{(*)}\mathrm{diag}(0,\Delta E_{21},\Delta E_{31})U^{\dagger(T)} + \mathrm{diag}(\pm A,0,0)\right]\begin{pmatrix}\nu_e(\bar{\nu}_e)\\ \nu_\mu(\bar{\nu}_\mu)\\ \nu_\tau(\bar{\nu}_\tau)\end{pmatrix}$$

$$= \tilde{U}^{(\pm)}\mathrm{diag}(\tilde{E}_1^{(\pm)},\tilde{E}_2^{(\pm)},\tilde{E}_3^{(\pm)})\tilde{U}^{(\pm)\dagger}\begin{pmatrix}\nu_e(\bar{\nu}_e)\\ \nu_\mu(\bar{\nu}_\mu)\\ \nu_\tau(\bar{\nu}_\tau)\end{pmatrix}$$

If the matter effect is constant,
this equation can be easily solved.

The contribution of each energy-dependent function to the oscillation probability





$$P(\nu_{\mu\to e}) = 4\left|U_{\mu 3}U_{e3}^*\right|^2 \sin^2(\Delta_{31}) + 4\mathrm{Re}\left[U_{\mu 3}U_{e3}^*U_{\mu 2}^*U_{e2}\right]\Delta_{21}\sin(2\Delta_{31})$$
$$-4\mathrm{Re}\left[U_{\mu 2}U_{e2}^*U_{\mu 1}^*U_{e1}\right]\Delta_{21}^2 - 8\mathrm{Im}\left[U_{\mu 3}U_{e3}^*U_{\mu 2}^*U_{e2}\right]\Delta_{21}\sin^2(\Delta_{31})$$

$$= C_1 \cdot \sin^2(\Delta_{31}) + C_2 \cdot \Delta_{21}\sin(2\Delta_{31})$$
$$+C_3 \cdot \Delta_{21}^2 + C_4 \cdot \Delta_{21}\sin^2(\Delta_{31})$$

Background subtraction

At the HK, in principle, ν_μ and $\bar{\nu}_\mu$ are distinguished by neutron tagging method.
We can define the oscillation probability $P(\nu_e \to \nu_\mu)$ as

$$P\left(\nu_e \to \nu_\mu\right) = \frac{1}{\kappa} \frac{\left(\kappa N_{\text{far}}^{\nu_e \to \nu_\mu} + (1-\kappa) N_{\text{far}}^{\bar{\nu}_\mu \to \bar{\nu}_\mu}\right) - (1-\kappa) N_{\text{far}}^{\bar{\nu}_\mu \to \bar{\nu}_\mu}\Big|_{\text{T2HK}}}{N_{\text{near}}^{\nu_e \to \nu_e}}$$

$$\kappa \equiv \frac{1 + C_{\text{id}}}{2}\,, C_{\text{id}}\text{: charge identification efficiency}$$

In the case of $C_{\text{id}} = 0.0$, we just do not perform the charge identification analysis and simply add the background events, and subtract the estimated amount by using the T2HK data, i.e. we take $\kappa = 1$ and $1 - \kappa = 1$.

Probability of $\nu_e \to \nu_\mu$

At the HK, in principle, ν_μ and $\bar{\nu}_\mu$ are distinguished by neutron tagging method.
We can define the oscillation probability $P(\nu_e \to \nu_\mu)$ as

Perfect charge identification

$$P(\nu_e \to \nu_\mu) = \frac{N_{\text{far}}^{\nu_e \to \nu_\mu}}{N_{\text{near}}^{\nu_e \to \nu_e}}$$

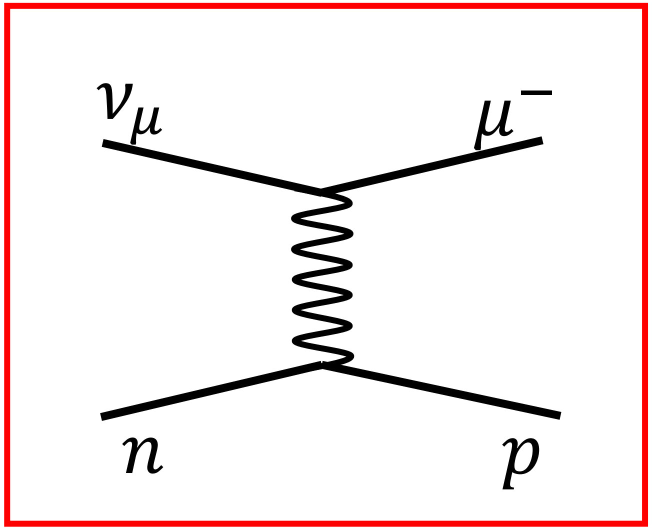
No charge identification

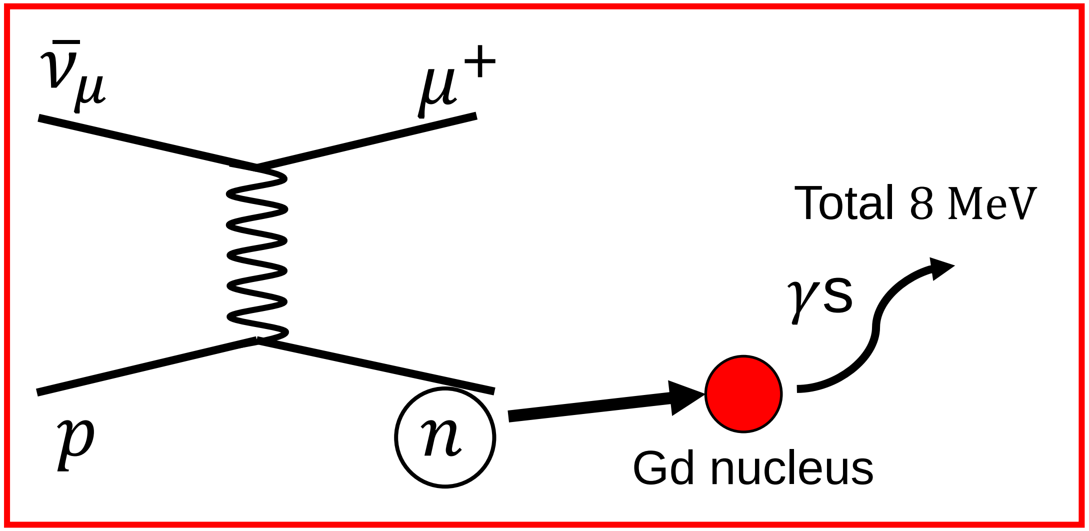
$$P(\nu_e \to \nu_\mu) = \frac{\left(N_{\text{far}}^{\nu_e \to \nu_\mu} + N_{\text{far}}^{\bar{\nu}_\mu \to \bar{\nu}_\mu}\right) - N_{\text{far}}^{\bar{\nu}_\mu \to \bar{\nu}_\mu}\Big|_{\text{T2HK}}}{N_{\text{near}}^{\nu_e \to \nu_e}}$$

Neutron tagging

SK-Gd : efficiency ~ 70%
Hyper-K : efficiency ≥ 70%

R. Akutsu. Ph.D thesis, Tokyo University, 2019.





Derivation of χ^2

$$\chi^2 \equiv \sum_j \frac{\left(N_{\mathrm{far},j}^{\mathrm{obs}} - N_{\mathrm{far},j}^{\mathrm{exp}}\right)^2}{\left(\Delta N_j^{\mathrm{obs}}\right)^2}$$

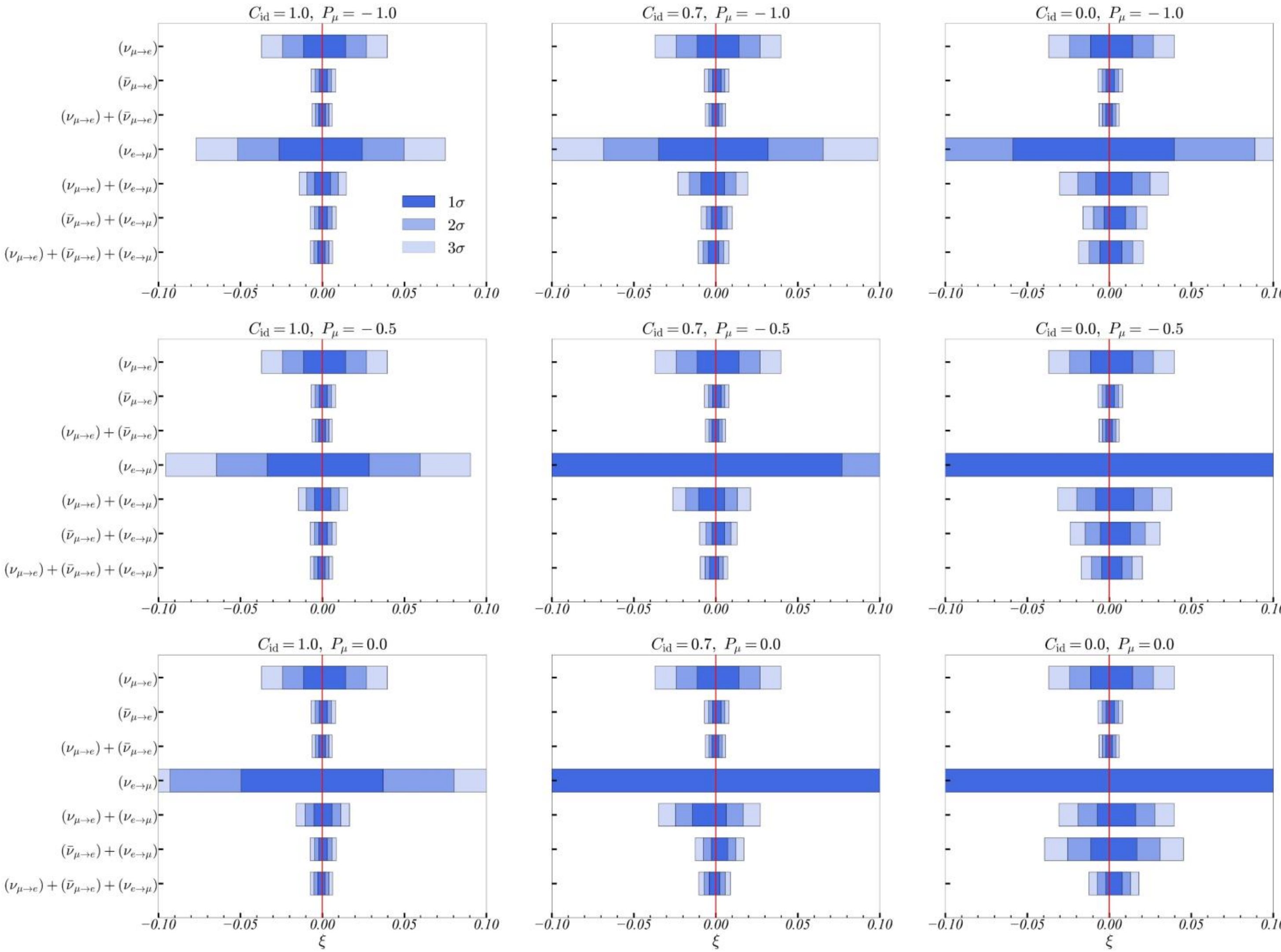
$$= \sum_j \frac{\left(N_{\mathrm{far},j}^{\mathrm{obs}} - \widetilde{N}_{\mathrm{near},j}^{\mathrm{obs}} \cdot P\left(E_j\right)\right)^2}{\left(\Delta N_{\mathrm{far},j}^{\mathrm{obs}}\right)^2 + \left(\Delta \widetilde{N}_{\mathrm{near},j}^{\mathrm{obs}} \cdot P_j^{\mathrm{obs}}\right)^2}$$

$$= \sum_j \frac{\left[\frac{N_{\mathrm{far},j}^{\mathrm{obs}}}{\widetilde{N}_{\mathrm{near},j}^{\mathrm{obs}}} - (\mathcal{B}\boldsymbol{C})_j\right]^2}{\left(P_j^{\mathrm{obs}}\right)^2 \cdot \left[\left(\frac{\Delta N_{\mathrm{far},j}^{\mathrm{obs}}}{N_{\mathrm{far},j}^{\mathrm{obs}}}\right)^2 + \left(\frac{\Delta N_{\mathrm{near},j}^{\mathrm{obs}}}{N_{\mathrm{near},j}^{\mathrm{obs}}}\right)^2\right]}$$

$$= \sum_j \left[\frac{P^{\mathrm{obs}}\left(E_j\right) - \sum_{k=1}^{4} C_k \cdot B_k}{\Delta P\left(E_j\right)}\right]^2$$

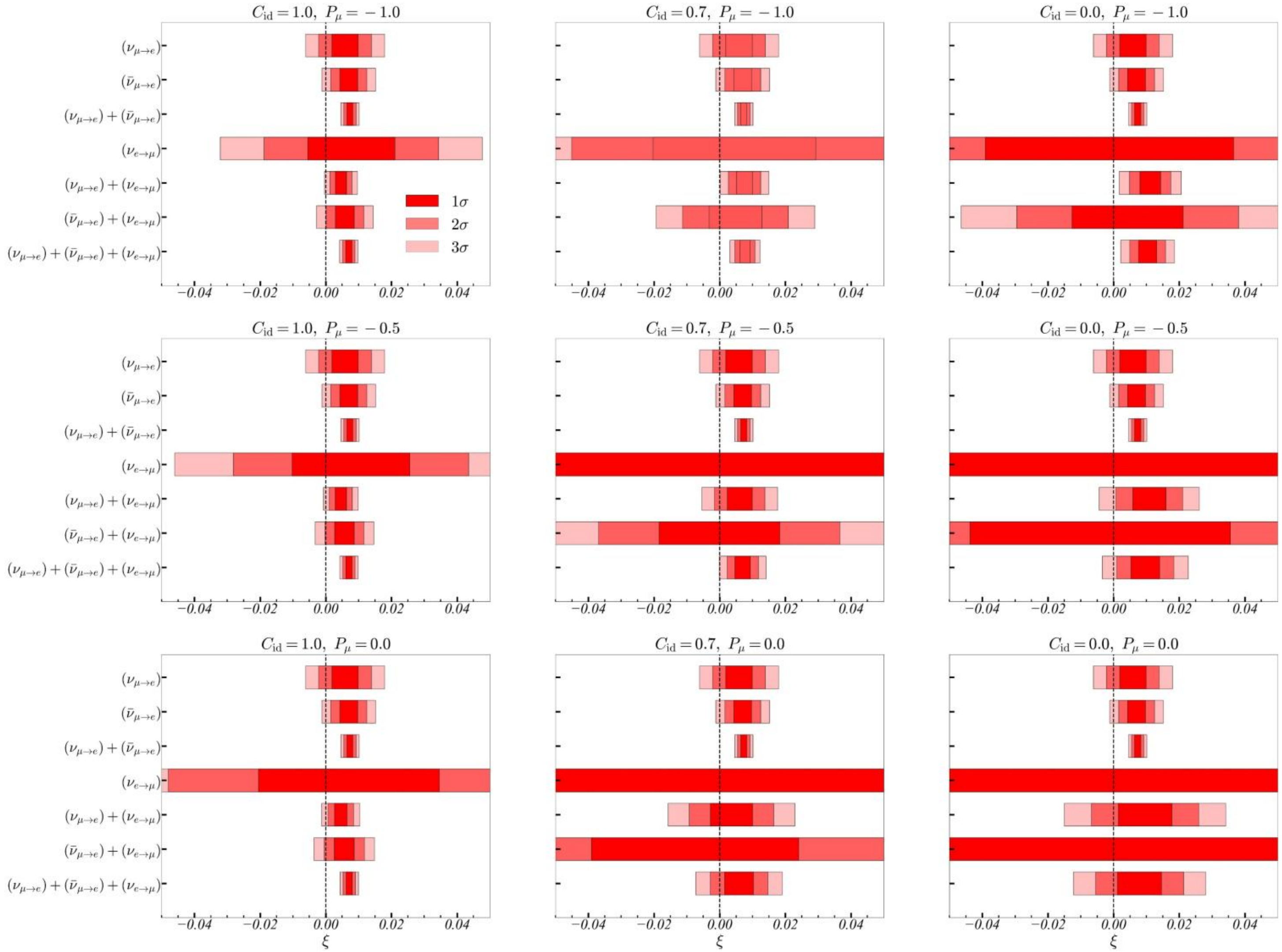
Unitarity Test

Fitting 3-gen model to 3-gen events



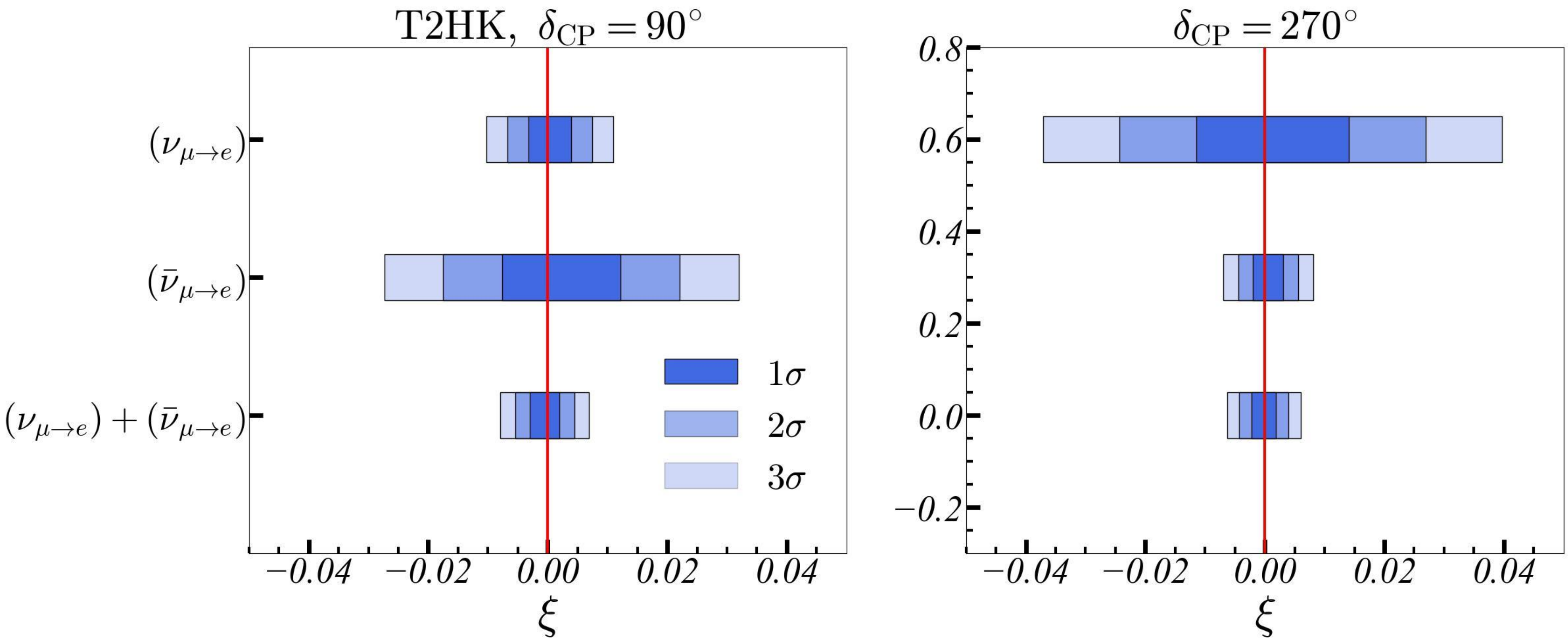
Unitarity Test

Fitting 3-gen model to 4-gen events



Unitarity Test

Fitting 3-gen model to 3-gen events



Opposite behavior arises from

$\delta \to -\delta \quad \Longleftrightarrow \quad (\nu_{\mu\to e}) \to (\bar\nu_{\mu\to e})$

Which uncertainty is smaller depends on δ

Virtual-experiments

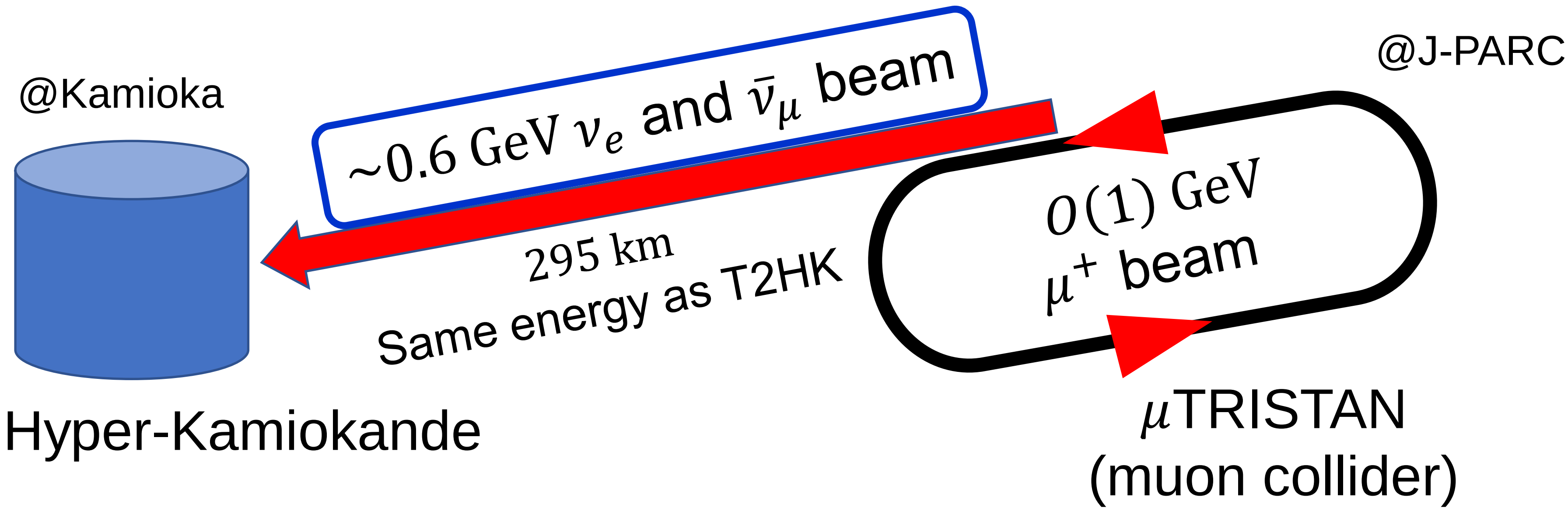
Assume a binomial distribution $B(n, p)$,
since the observed number of events is an integer

$$B(n,p) \begin{cases} n\text{: Number of non-oscillated events} \\ p\text{: Oscillation probability} \end{cases}$$

n, p can be assigned to each energy bin

→ the virtual-experiments are randomly generated according to the binomial distribution in each bin

Future neutrino factory



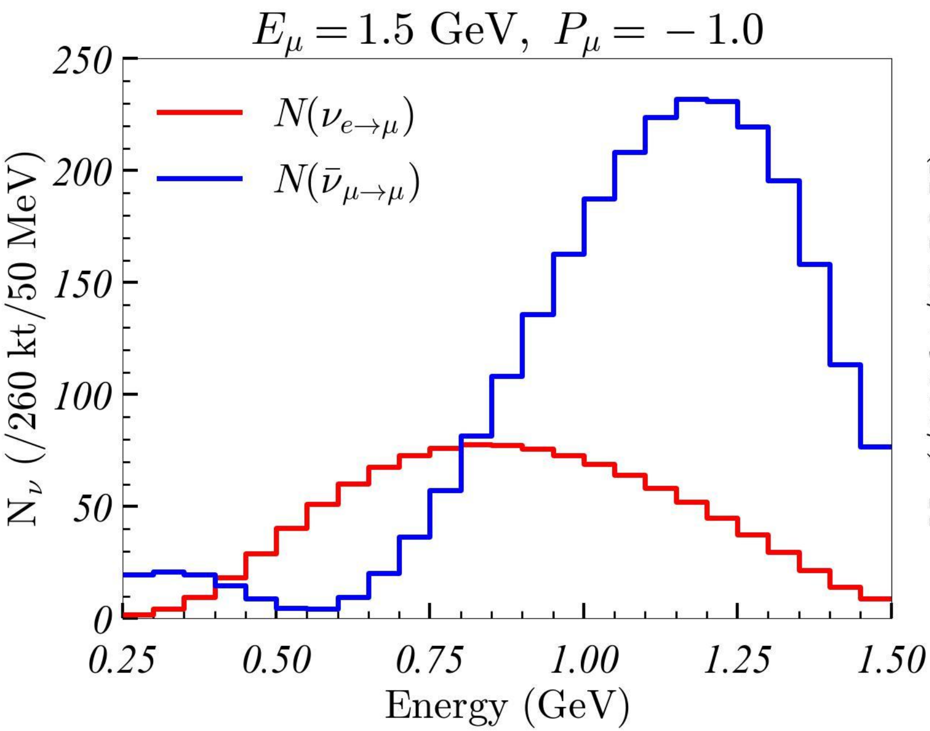
Number of events

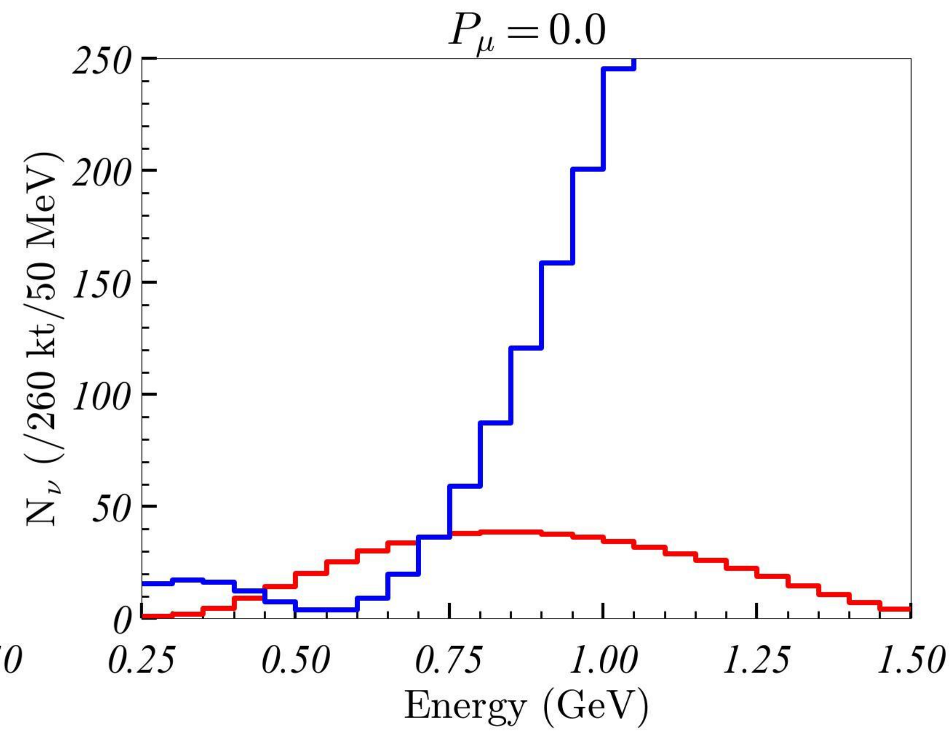
$$N_j^{\nu_e \to \nu_\mu} = \int_{E_j}^{E_{j+1}} \frac{dE_\nu}{E_\mu} \times \frac{12 N_\mu \cdot V \cdot n_N}{\pi L^2} \times \gamma^2 \left(\frac{E_\nu}{E_\mu}\right)^2 \times \left[\left(1 - \frac{E_\nu}{E_\mu}\right) - P_\mu \left(1 - \frac{E_\nu}{E_\mu}\right)\right]$$
$$\times P_j^{\nu_e \to \nu_\mu}(E_\nu) \times \sigma_{\nu_\mu}(E_\nu)$$

$N_j^{\nu_e \to \nu_\mu}$: number of events
N_μ : number of muons
V : detector volume
n_N : number of density of the nucleon in water
$\gamma = \frac{E_\mu}{m_\mu}$: boost factor
E_ν : neutrino beam energy
E_μ : muon beam energy
P_μ : polarization of anti-muon beam
σ_{ν_μ} : cross section

Neutrino flux in neutrino factory

Total number of muons (which decay toward HK) is 10^{22}





Neutrino flux in T2HK

2.7×10^{22} POT $(\nu : \bar{\nu} = 1 : 3)$
calculated based on K. Abe et al. 2018